International Journal of Research in Advent Technology, Vol.5, No.9, September 2017
E-ISSN: 2321-9637
Available online at www.ijrat.org

Solution for Nonlinear Riccati Differential Equations by Variational Iteration Method

Monika Rani[*,1], Harbax Singh Bhatti[2], Vikramjeet Singh[3]
[1]*Research Scholar, I K Gujral Punjab Technical University, Kapurthala, Punjab, India*
[2]*Baba Banda Singh Bahadur Engineering College, Fatehgarh Sahib, Punjab, India*
[3]*I K Gujral Punjab Technical University, Kapurthala, Punjab, India*
**Email: gargnitk24@gmail.com*

Abstract- In this manuscript, we represent the quadratic nonlinear Riccati differential equation and analyze it using variational iteration method. The method provides simple and fast convergent solution without any restrictive assumptions. A comparison of obtained solution with exact one has been shown geometrically. Some examples of non linear Riccati differential equation have also been shown to validate the consistency and effectiveness of these methods.

Index Terms: Riccati differential equation; Non Perturbation Methods; Variational iteration method (VIM).

1. INTRODUCTION

In recent years, due to the rapid development of non linear science, the researchers have shown their interest in analytical techniques to solve a wide variety of linear, nonlinear ordinary and partial differential equations [1]. A number of techniques have been proposed to solve the non linear equations, but these analytical techniques have their own assumptions, deficiencies and limitations. However, some methods provide readily verifiable and rapidly convergent approximate solution with less computation such as

(i) Adomian Decomposition Method,
(ii) Homotopy Perturbation Method and
(iii) He's Variational Iteration Method.

Using these methods, we get analytical approximation or an approximated solution to a relatively broad category of nonlinear (and stochastic) equations [2-4].

2. VARIATIONAL ITERATION METHOD (VIM)

VIM was first proposed by Ji Huan He in 1999 for solving linear, non linear, initial and boundary value problems. It is worth mentioning that the origin of VIM is traced back to Innokuti, Sekine and Mura, but the real potential of the VIM was explored by He [5]. In this method, a correction functional is constructed by a general lagrange multiplier, which can be identified via variational theory so that each iteration improves the accuracy of the solution [6, 7]. VIM does not depend on small parameter and Adomian polynomials which are essential in perturbation method and ADM respectively. This method is also more powerful than ADM due to an ease of initial guess of unknown parameter. These properties of VIM make it widely applicable in handling non-linear problems.

2.1 Brief description of VIM

First, consider a general differential equation

$$Lu + Nu = H(x) \tag{1}$$

Where H(x) represents an inhomogeneous term, L and N are linear and non linear operators respectively. Now, consider a correction functional as

$$u_{k+1}(x) = u_k(x) + \int_0^x \lambda \left(Lu_k(\zeta) + N\tilde{u}_k(\zeta) - H(\zeta) \right) d\zeta \tag{2}$$

where λ is a Lagrange multiplier, k represents the kth approximation, $\tilde{u}_k$ can be shown as a restricted variation i.e. $\delta \tilde{u}_k = 0$. The successive approximation u_{k+1}, $k \geq 0$ of the solution u can be readily obtained by the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by

$$u = \lim_{k\to\infty} u_k \tag{3}$$

2.2 Numerical Example on VIM

Consider a non linear Riccati differential equation,

$\frac{du}{dx} = -2 - u + u^2$, with initial condition $u(0) = 2$

having exact solution u(x) = 2.

To solve the Riccati differential equation using VIM, the correction functional will be

$u_{k+1}(x) = u_k(x) + \int_0^x \lambda\, (u_k'(\zeta) - \tilde{u}_k^2(\zeta) + 2 + u_k(\zeta))\, d\zeta, \quad k \geq 0$ (4)

Taking variation w. r. t. independent variable u_k and $\delta(\tilde{u}_k^2) = 0$, then

$\delta u_{k+1}(x) = \delta u_k(x) + \delta \int_0^x \lambda\, (u_k'(\zeta) - \tilde{u}_k^2(\zeta) + 2 + u_k(\zeta))\, d\zeta$ (5)

$\delta u_{k+1}(x) = \delta u_k(x) + [\lambda\, \delta u_k(\zeta)]_{\zeta = x} - \int_0^x \lambda'\, \delta u_k(\zeta)\, d\zeta$ (6)

Which gives the stationary conditions,

$[1 + \lambda(\zeta)]_{\zeta=x} = 0,$ (7)

$[\lambda'(\zeta)]_{\zeta=x} = 0$ (8)

Equations (7) and (8) are called as Lagrange Euler equation and Natural boundary condition respectively. Using these equations, the Lagrange multiplier can be calculated as $\lambda = -1$. Then, equation (4) becomes

$u_{k+1}(x) = u_k(x) - \int_0^x (u_k'(\zeta) - u_k^2(\zeta) + 2 + u_k(\zeta))\, d\zeta,$ (9)

Using initial condition, $u_0 = 2$ and the successive approximation can be calculated after putting the value of u_0 in equation (9), as

$u_0 = u_1 = u_2 = \ldots\ldots\ldots = u_k = 2$ (10)

Hence, the exact solution is

$u(x) = \lim_{k\to\infty} u_k(x) = 2$ (11)

2.3 Another Example of Riccati Equation:

Consider an equation

$y'(t) + cost\; y(0.8t) - Sint\; y^2(t+2) = cost[1 + Sin(0.8t)] - Sint\; Sin^2(t+2),$
$0 \leq t \leq 1$ (12)

With initial condition y(0) = 0, having exact solution is

y(t) = Sint (13)

Here, $Ly = \frac{\partial y}{\partial t}$, (14)

$Ny = -Sint\; y^2(t+2)$ (15)

$H(t) = Cost(1 + Sin\,(0.8t)) - Sint\; Sin^2(t+2) - Cost\; y(0.8t)$ (16)

The correctional function is

$y_{k+1}(t) = y_k(t) + \int_0^t \lambda(\zeta)\left[y_k'(\zeta) - \operatorname{Sin}\zeta\, \tilde{y}_k^2(\zeta + 2) - H(\zeta)\right] d\zeta$ (17)

Taking variation w. r. t. independent variable y_k with $\delta N(\tilde{y_k}) = 0$, gives

$\lambda = -1$ (18)

Now, equation (17) can be rewritten as

$y_{k+1}(t) = y_k(t) - \int_0^t \left[y_k'(\zeta) - \operatorname{Sin}\zeta\, y_k^2(\zeta + 2) - H(\zeta)\right] d\zeta$ (19)

Using initial condition, $y_0 = 0$ and the first approximation y_1 can be calculated after putting the value of y_0 in equation (19), as

$y_1 = Sin\,t - \frac{1}{3.6}\,Cos\,(1.8t) + \frac{1}{0.4}\,Cos\,(0.2t) + 0.5\,Cos\,t - \frac{1}{12}\,Cos\,(3t+4) + 0.25\,Cos\,(t+4) - 2.613282$ (20)

The graph of exact solution and VIM solution of this example has been shown in figure 1. It is found that absolute error of proposed method is between $7.19\text{x}10^{-5}$ to $3.81\text{x}10^{-7}$.

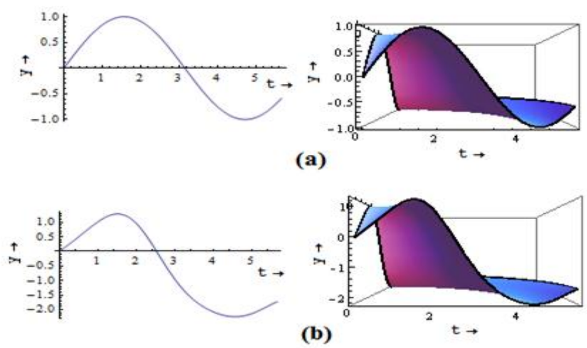

Figure 1: The graphical representation of Riccati solution: (a) Exact solution, (b) VIM solution

It is revealed that the uppermost accuracy has been achieved in the first iteration so no need for further computations. It is also shown that our solution is very close to exact solution. After comparing the results, it has been proven that our VIM solution is highly accurate with fast convergence.

3. CONCLUSION

In this paper, we have provided readily verifiable and rapidly convergent approximate solution of quadratic Riccati differential equation with less computation using He's Variational Iteration Method. The proposed approach has shown high degree of accuracy as contrasted to existing techniques. Using these methods, we get an approximated solution to a relatively broad category of nonlinear (and stochastic) equations.

4. ACKNOWLEDGEMENT

The authors are highly obliged to I. K. Gujral Punjab Technical University, Kapurthala for providing a platform to accomplish this research.

REFERENCES

[1] S. T. Mohyud-Din, M. A. Noor, "Homotopy Perturbation Method for Solving Partial Differential Equations", Zeitschrift für Naturforschung A., vol. 64, no. 3-4, pp. 157-170, 2014.

[2] H. Cruz, D. Schuch, O. Castaños, O. Rosas-Ortiz, "Time-evolution of quantum systems via a

complex nonlinear Riccati equation. II. Dissipative systems", Annals of Physics, vol. 373, pp. 609–630, 2016.

[3] S. Mukherjee, B. Roy, "Solution of Riccati Equation with Variable Co-efficient by Differential Transform Method", International Journal of Nonlinear Science, vol.14, no.2, pp.251-256, 2012.

[4] A. A. Bahnasawi, M. A. El-Tawil, A. Abdel-Naby, "Solving Riccati differential equation using Adomian's decomposition method", Appl. Math. Comput., vol. 157, pp. 503-514, 2004.

[5] J. H. He, "Homotopy perturbation method: a new non linear analytical technique", Applied Mathematics and Computation, vol. 135, pp. 73-79, 2003.

[6] S. T. Mohyud-Din, M. A. Noor, "Solving Schrödinger Equations by Modified Variational Iteration Method", World Applied Sciences Journal, vol. 5, no. 3, pp. 352-357, 2008.

[7] S. Zhang, "Exp-function method for constructing explicit and exact solutions of a lattice equation", Applied Mathematics and Computation, vol. 199, no. 1, pp. 242-249, 2008.